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Second-order fourth-degree Painlevé-type equations

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Abstract

Transformations that involve a Fuchsian-type equation are used to obtain one-to-one correspondence between the Painlevé I–IV equations and certain second-order fourth-degree Painlevé-type equations.

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1. Introduction

An ordinary differential equation is said to be of Painlevé type, or to have the Painlevé property, if the only movable singularities of its solutions are poles. The only first-order first-degree differential equation that is of Painlevé type is the Riccati equation. First-order and higher-degree differential equations of Painlevé type have been studied by Briot, Bouquet and Fuchs (see [1]). Briot and Bouquet have classified all binomial equations of the form

$$(v')^m + F(z, v) = 0, \tag{1.1}$$

where F is a rational function of v and locally analytic in z and m is a positive integer, that are of Painlevé type. Fuchs (see [1]), Ince [1] and Chalkley [9] study Painlevé-type equations of the form

$$a_1(z, v)(v')^n + a_2(z, v)(v')^{n-1} + \cdots + a_{n-1}(z, v)v' + a_n(z, v) = 0 \tag{1.2}$$

where $a_j(z, v)$ are assumed to be polynomials in v whose coefficients are analytic functions of z and $a_1(z, v) \neq 0$. The necessary and sufficient conditions for these equations to be of Painlevé type are given by the Fuchs theorem. The Fuchs theorem shows that, apart from other conditions, the irreducible form of the first-order and second-degree Painlevé-type equation is

$$\begin{aligned} a_1(z)(v')^2 + [a_2(z)v^2 + a_3(z)v + a_4(z)]v' + a_5(z)v^4 + a_6(z)v^3 + a_7(z)v^2 \\ +a_8(z)v + a_9(z) \end{aligned} \tag{1.3}$$

where $a_j(z)$, $j = 1, 2, \ldots, 9$ are analytic functions of z and $a_1(z) \neq 0$. Let

$$F(v) := A_0v^4 + A_1v^3 + A_2v^2 + A_3v + A_4 \tag{1.4}$$

where

$$\begin{aligned} &A_0 = 4a_1a_5 - a_2^2 \qquad A_1 = 4a_1a_6 - 2a_2a_3 \\ &A_2 = 4a_1a_7 - 2a_2a_4 - a_3^2 \\ &A_3 = 4a_1a_8 - 2a_3a_4 \qquad A_4 = 4a_1a_9 - a_4^2. \end{aligned} \tag{1.5}$$

It is known that when $F(v) \neq 0$, there are unique polynomials $F_1(v)$ and $F_2(v)$ such that

$$F(v) = A(z)F_1(v)[F_2(v)]^2 \tag{1.6}$$

where $A(z)$ is an analytic function and $F_1(v)$ has no multiple roots. In [9], it was shown (theorem 6.2) that equation (1.3) is of Painlevé type if and only if the following conditions hold:

$$\begin{aligned} &\text{(i)} \qquad F_1(v) \text{ divides } G_1(v) := (a_2v^2 + a_3v + a_4)\frac{\partial F_1}{\partial v} - 2a_1\frac{\partial F_1}{\partial z} \\ &\text{(ii)} \qquad A_0 = 0 \text{ and } A_1 \neq 0 \text{ imply } a_2 = 0 \\ &\text{(iii)} \qquad A_0 = A_1 = A_2 = 0 \text{ and } A_3 \neq 0 \text{ imply } a_2 = 0. \end{aligned} \tag{1.7}$$

It was also shown (corollary 6.3) that the special case

$$G(v) := (a_2v^2 + a_3v + a_4)\frac{\partial F}{\partial v} - 2a_1\frac{\partial F}{\partial z} = 0 \tag{1.8}$$

is of Painlevé type.

The best known second-order first-degree Painlevé-type equations are the so-called Painlevé equations, PI, PII, ..., PVI [1], discovered by Painlevé and his school. They classified all equations of the form

$$v'' = F(z, v, v') \tag{1.9}$$

where F is rational in v' and v and locally analytic in z. They found that there are 50 such equations. The Painlevé equations, PI, PII, ..., PVI, are the only irreducible ones and define new transcendents. The other 44 equations are either solvable in terms of the known functions or can be transformed into one of the six equations.

Painlevé-type equations of the second order and degree two or higher have been studied in [2–4]. In [4], all binomial-type equations of the form

$$(v'')^m = F(z, v, v') \qquad m \geqslant 3 \tag{1.10}$$

were classified. It turns out that there are two second-order fourth-degree Painlevé-type equations, labelled as BP-IX and BP-X, of the form of (1.10). BP-IX and BP-X were solved in terms of elliptic functions or the special case of the second Painlevé transcendent.

Fokas and Ablowitz [8] developed an algorithmic method to investigate the transformation properties of the Painlevé equations. However, certain second-order second-degree equations of Painlevé-type equations related to PIII and PVI were also discussed. They used the transformation

$$u = \frac{v' + av^2 + bv + c}{dv^2 + ev + f} \tag{1.11}$$

where a, b, c, d, e, f are functions of z only. The transformation (1.11) is the only transformation that is linear in v' and preserves the Painlevé property. The aim was to find a, b, c, d, e, f such that (1.11) defines a one-to-one invertible map between solutions v of the Painlevé equations and solutions u of some second-order second-degree equations of Painlevé type. In [5, 6] the same method was used to derive all second-order second-degree equations of Painlevé type related to PI–PVI by transformations of the form (1.11).

As an extension to the method of Fokas and Ablowitz, one may replace (1.11) by a transformation of the form

$$u = \frac{(v')^2 + (a_2v^2 + a_1v + a_0)v' + b_4v^4 + b_3v^3 + b_2v^2 + b_1v + b_0}{(c_2v^2 + c_1v + c_0)v' + d_4v^4 + d_3v^3 + d_2v^2 + d_1v + d_0} \tag{1.12}$$

where a_j, b_k, c_j, d_k, $j = 0, 1, 2$, $k = 0, 1, 2, 3, 4$, are functions of z. Let $A_j := c_ju - a_j$, $B_k := d_ku - b_k$, $j = 0, 1, 2$, $k = 0, 1, 2, 3, 4$. Then the transformation (1.12) preserves the Painlevé property if the equation

$$(v')^2 = (A_2v^2 + A_1v + A_0)v' + B_4v^4 + B_3v^3 + B_2v^2 + B_1v + B_0 \tag{1.13}$$

is of Painlevé type. Comparing this equation with (1.3) one finds

$$\begin{aligned} F(v) = -[&(4B_4 + A_2^2)v^4 + 2(2B_3 + A_1A_2)v^3 + (4B_2 + A_1^2 + 2A_0A_2)v^2 \\ &+2(2B_1 + A_0A_1)v + (4B_0 + A_0^2)]. \end{aligned} \tag{1.14}$$

Thus equation (1.13) is of Painlevé type if it satisfies the following conditions:

$$\begin{array}{ll} \text{(i)} & F_1(v) \text{ divides } G_1(v) := -(A_2v^2 + A_1v + A_0)\dfrac{\partial F_1}{\partial v} - 2\dfrac{\partial F_1}{\partial z} \\ \text{(ii)} & 4B_4 + A_2^2 = 0 \text{ and } 2B_3 + A_1A_2 \neq 0 \text{ imply } A_2 = 0 \\ \text{(iii)} & 4B_4 + A_2^2 = 2B_3 + A_1A_2 = 4B_2 + A_1^2 + 2A_0A_2 = 0 \\ & \text{and } 2B_1 + A_0A_1 \neq 0 \text{ imply } A_2 = 0 \end{array} \tag{1.15}$$

where $F_1(v)$ is the unique polynomial defined by equation (1.6). In [7], the transformation (1.12) is used to obtain one-to-one correspondence between solutions v of PI–PVI equations and solutions u of certain second-order second-degree equations of Painlevé type.

In this paper, the transformation (1.12) will be used to obtain one-to-one correspondence between solutions v of PI–PIV equations and solutions u of certain second-order fourth-degree equations of Painlevé type. For the sake of simplicity, we use the transformation (1.12) subject to the constraint

$$G(v) := -(A_2v^2 + A_1v + A_0)\frac{\partial F}{\partial v} - 2\frac{\partial F}{\partial z} = 0. \tag{1.16}$$

The procedure is as follows: let $v(z)$ be a solution of one of the Painlevé equations, which has the general form

$$v'' = P_2(v, z)(v')^2 + P_1(v, z)v' + P_0(v, z) \tag{1.17}$$

and let $u(z)$ be given by the transformation (1.12). The aim is to choose a_j, b_k, c_j and d_k such that $u(z)$ is a solution of a second-order fourth-degree equation of Painlevé type. To be more specific, differentiating the equation (1.13) and using (1.17) to replace v'' and (1.13) to replace $(v')^2$ one obtains

$$\Phi v' + \Psi = 0 \tag{1.18}$$

where

$$\begin{aligned} \Phi &= P_1 - 2A_2v - A_1 + 2P_2(A_2v^2 + A_1v + A_0) \\ \Psi &= 2P_2(B_4v^4 + B_3v^3 + B_2v^2 + B_1v + B_0) + \frac{1}{4}\frac{\partial F}{\partial v} + 2P_0 - (A_2'v^2 + A_1'v + A_0'). \end{aligned} \tag{1.19}$$

Now the aim is to choose a_j, b_k, c_j and d_k so that Φ and Ψ are identically zero and the constrained $G(v) = 0$ is reduced to a quadratic equation for v

$$A(u', u, z)v^2 + B(u', u, z)v + C(u', u, z) = 0. \tag{1.20}$$

Then, solving the equation (1.20) for v and substituting into equation (1.13) one obtains second-order fourth-degree Painlevé-type equations for u. If one reduces $G(v) = 0$ to a linear equation for v

$$A(u', u, z)v + B(u', u, z) = 0 \tag{1.21}$$

then substituting $v = -B/A$ into equation (1.13) gives second-order second-degree Painlevé-type equations for u.

It turns out that PV and PVI do not have a transformation of this type. In the case of PI, PII and PIV, there is only one choice of a_j, b_k, c_j and d_k that reduces $G(v) = 0$ to a quadratic equation for v while in the case of PIII there are two choices. In each case, a second-order fourth-degree Painlevé-type equation will be obtained. Some of the second-order fourth-degree Painlevé-type equations will be used to rederive some known discrete Lie-point symmetries of PII and PIII.

PI and a special case of PIII only have transformations such that $G(v) = 0$ can be reduced to a linear equation for v. The case of PI has been considered in [7]. The case of PIII will be investigated in this paper. Throughout this paper $'$ denotes the derivative with respect to z and $\dot{}$ denotes the derivative with respect to x.

2. Painlevé I

The first Painlevé equation, PI, is

$$v'' = 6v^2 + z. \tag{2.1}$$

For PI, equation (1.19) takes the form of

$$\begin{aligned}\Phi &= -(2A_2v + A_1)\\ \Psi &= -[(A_2^2 + 4B_4)v^3 + \tfrac{1}{2}(6B_3 + 3A_1A_2 + 2A_2' - 24)v^2\\ &\quad + \tfrac{1}{2}(4B_2 + 2A_0A_2 + A_1^2 + 2A_1')v + \tfrac{1}{2}(2B_1 + A_0A_1 + 2A_0' - 4z)].\end{aligned} \tag{2.2}$$

To make $\Phi = \Psi = 0$, one should choose $A_2 = A_1 = B_4 = B_2 = 0$, $A_0 = -a_0$, $B_3 = 4$ and $B_1 = 2z - A_0'$. One can always absorb b_0 and d_0 in u by a proper Möbius transformation. Hence, without loss of generality, one can set $B_0 = u$. With these choices one obtains

$$F(v) = -[16v^3 + 4(2z - A_0')v + 4u + A_0^2] \tag{2.3}$$

and hence

$$G(v) = 8[6A_0v^2 + (2 - A_0'')v + u' + zA_0]. \tag{2.4}$$

Therefore, if $A_0 \neq 0$, the equation $G(v) = 0$ gives the following quadratic equation for v:

$$6A_0v^2 - (A_0'' - 2)v + u' + zA_0 = 0. \tag{2.5}$$

The equation (1.13) becomes

$$(v')^2 = A_0v' + 4v^3 - (A_0' - 2z)v + u. \tag{2.6}$$

Let $u(z) = \frac{3}{2}y(x) + q(x)$, $z = r(x)$ and $A_0(z) = e(x)$, where $r(x) = -\int \frac{\mathrm{d}x}{e(x)}$, $\dot{q} - r + \frac{1}{24e^2}(e^2\ddot{e} + e\dot{e}^2 - 2)^2 = 0$. Then, following the procedure discussed in the introduction yields the following second-order fourth-degree Painlevé-type equation for $y(x)$:

$$\left[(\ddot{y})^2 - \frac{4}{e^2}\dot{y}(6\dot{f}\dot{y} + 6y + g)\right]^2 = 16\dot{y}\left[(\ddot{f} + 1)\ddot{y} - \frac{2}{e^2}\dot{y}(\dot{y} + 3\dot{f}^2 + 6f)\right]^2 \tag{2.7}$$

where

$$f(x) = \tfrac{1}{6}(e\dot{e} + 2r) \qquad g(x) = 2(\dot{f})^3 + 12f\dot{f} + 4q + e^2 - e^2(\ddot{f} + 1)^2. \tag{2.8}$$

Equations (2.5) and (2.6) give one-to-one correspondence between solutions $v(z)$ of PI and solutions $y(x)$ of (2.7).

If $A_0 = 0$, then $G(v) = 0$ reduces to a linear equation for v. Solving that equation for v and substituting in equation (2.6), one obtains a second-order second-degree equation for u [7].

3. Painlevé II

The second Painlevé equation, PII, is

$$v'' = 2v^3 + zv + \alpha. \tag{3.1}$$

For PII, Φ and Ψ take the form of

$$\begin{aligned}\Phi &= -(2A_2 v + A_1)\\ \Psi &= -[(A_2^2 + 4B_4 - 4)v^3 + \tfrac{1}{2}(6B_3 + 3A_1A_2 + 2A_2')v^2\\ &\quad + \tfrac{1}{2}(4B_2 + 2A_0A_2 + A_1^2 + 2A_1' - 4z)v + \tfrac{1}{2}(2B_1 + A_0A_1 + 2A_0' - 4\alpha)].\end{aligned} \tag{3.2}$$

Setting $\Phi = \Psi = 0$, one obtains $A_2 = A_1 = B_3 = 0$, $A_0 = -a_0$, $B_4 = 1$, $B_2 = z$ and $B_1 = -(A_0' - 2\alpha)$. One can always absorb b_0 and d_0 in u by a proper Möbius transformation. Hence, without loss of generality, one can set $B_0 = u$. The equation $G(v) = 0$ reads

$$2A_0v^3 + v^2 + (zA_o - A_0'')v + u' + \alpha A_0 = 0. \tag{3.3}$$

If $A_0 \neq 0$, then in order to reduce (3.3) to a quadratic equation for v one may try to write it as

$$(v - g)(Av^2 + Bv + C) = 0 \tag{3.4}$$

where g is a function of z only. To achieve this aim, $g(z)$ must be chosen so that

$$u' + \alpha A_0 + g[g(2gA_0 + 1) + zA_0 - A_0''] = 0 \tag{3.5}$$

which is not possible. Therefore, the only way to reduce (3.3) to a quadratic equation for v is to set $A_0 = 0$.

Setting $A_0 = 0$, one obtains the following quadratic equation for v:

$$v^2 + u' = 0. \tag{3.6}$$

As a result of these choices the equation (1.13) becomes

$$(v')^2 = v^4 + zv^2 + 2\alpha v + u. \tag{3.7}$$

If $\alpha \neq 0$, then the equations (3.6) and (3.7) give one-to-one correspondence between solutions $v(z)$ of PII and solutions $u(z)$ of the following second-order fourth-degree Painlevé-type equation:

$$[(u'')^2 + 4(u')^3 - 4z(u')^2 + 4uu']^2 = -64\alpha^2(u')^3. \tag{3.8}$$

One can easily note that α and $-\alpha$ give the same value of u, that is, $u(z; \alpha) = u(z; -\alpha)$. Thus, using equation (3.6), one obtains $[v(z; \alpha)]^2 = [v(z; -\alpha)]^2$. Since $v(z; \alpha) \neq v(z; -\alpha)$, one obtains the well known discrete Lie-point symmetry of PII $v(z; \alpha) = -v(z; -\alpha)$ [8].

When $\alpha = 0$, the equation (3.8) reduces to the following second-order second-degree Painlevé-type equation:

$$(u'')^2 + 4(u')^3 - 4z(u')^2 + 4uu' = 0. \tag{3.9}$$

The change of variable $z = -\frac{1}{\sqrt[3]{2}}x$, $u(z) = -\sqrt[3]{2}y(x)$ transforms equation (3.9) into the special case ($\lambda_1 = 0$) of the equation

$$(\ddot{y})^2 = -4(\dot{y})^3 - 2\dot{y}(x\dot{y} - y) + \lambda_1. \tag{3.10}$$

In [3], equation (3.10) was labelled as SD-I.d and solved in terms of the PII equation

$$\ddot{w} = 2w^3 + xw + \epsilon\left(\sqrt{2\lambda_1} - \tfrac{1}{2}\right) \qquad (3.11)$$

where $\epsilon = \pm 1$. Equation (3.10) and its solution were rederived in [7]. The relation between solutions $u(z)$ of equation (3.9) and solutions $y(x)$ of equation (3.10), with $\lambda_1 = 0$, implies the following one-to-one correspondence between solutions $v(z)$ of the PII equation (3.1) with $\alpha = 0$ and solutions $w(x)$ of the PII equation (3.11) with $\lambda_1 = 0$:

$$w = -\frac{\epsilon v'}{\sqrt[3]{2}v} \qquad \epsilon\dot{w} + w^2 + \tfrac{1}{2}x = \sqrt[3]{2}v^2 \qquad (3.12)$$

provided that $v \neq 0$. The Bäcklund transformation (3.12) was first obtained by Gambier [10]. When $v = 0$, one obtains the known fact [8] that the PII equation (3.11) with $\lambda_1 = 0$ has a one-parameter family of solutions characterized by

$$\epsilon\dot{w} + w^2 + \tfrac{1}{2}x = 0. \qquad (3.13)$$

4. Painlevé III

Let $v(z)$ be a solution of the PIII equation

$$v'' = \frac{1}{v}(v')^2 - \frac{1}{z}v' + \gamma v^3 + \frac{1}{z}(\alpha v^2 + \beta) + \frac{\delta}{v}. \qquad (4.1)$$

Then Φ and Ψ take the following forms:

$$\begin{aligned}\Phi &= \frac{1}{v}\left[\left(A_1 - \frac{2}{z}\right)v + 2A_0\right]\\ \Psi &= \frac{1}{v}\Bigg[(2\gamma - A_2^2 - 2B_4)v^4 + \left(\frac{2\alpha}{z} - B_3 - \frac{3}{2}A_1A_2 - A_2'\right)v^3\\ &\quad -\left(A_1' + \frac{1}{2}A_1^2 + A_0A_2\right)v^2 + \left(B_1 - \frac{1}{2}A_0A_1 - A_0' + \frac{2\beta}{z}\right)v + 2B_0 + 2\delta\Bigg].\end{aligned} \qquad (4.2)$$

Setting $\Phi = \Psi = 0$, one obtains $A_0 = 0$, $A_1 = \frac{2}{z}$, $A_2 = -a_2$, $B_4 = \gamma - \frac{1}{2}A_2^2$, $B_3 = \frac{2\alpha}{z} - \frac{3}{z}A_2 - A_2'$, $B_1 = -\frac{2\beta}{z}$ and $B_0 = -\delta$. Without loss of generality one may set $B_2 = u$. In this case, the equation $G(v) = 0$ gives

$$\begin{aligned}&A_2(4B_4 + A_2^2)v^3 + \left[2B_4' + A_2A_2' + 3A_2\left(B_3 + \frac{1}{z}A_2\right) + \frac{2}{z}(4B_4 + A_2^2)\right]v^2\\ &\quad +2\left[B_3' + \frac{3}{z}A_2 + \frac{1}{z}A_2' + \frac{3}{z}B_3 + 2A_2u\right]v + 2\left[u' + \frac{2}{z}u - \frac{\beta}{z}A_2\right] = 0.\end{aligned} \qquad (4.3)$$

Now, the aim is to reduce equation (4.3) to a quadratic equation for v. This can be achieved only by setting the coefficient of v^3 equal to zero. There are two cases: (1) $A_2 = 0$ and (2) $A_2^2 + 4B_4 = 0$.

Case 1. $A_2 = 0$. If $\gamma \neq 0$, then the quadratic equation for v reads

$$\gamma z v^2 + \alpha v + \tfrac{1}{4}(z^2u' + 2zu) = 0 \qquad (4.4)$$

and the equation (1.13) reads

$$(v')^2 = \frac{2}{z}vv' + \gamma v^4 + \frac{2\alpha}{z}v^3 + uv^2 - \frac{2\beta}{z}v - \delta. \qquad (4.5)$$

Let $z = e^{-x}$, $u(z) = \frac{1}{\gamma}e^{2x}[y(x) - \alpha^2 x]$. Then there is a one-to-one correspondence, given by (4.4) and (4.5), between solutions $v(z)$ of the PIII equation and $y(x)$ of the following second-order fourth-degree equation of Painlevé type:

$$\left\{(\ddot{y})^2 + 8\dot{y}\ddot{y} - \frac{1}{\gamma}\dot{y}[(\dot{y})^2 + 2(2y - f)\dot{y} + (4\alpha^2 y + g)]\right\}^2 = 64\dot{y}\left[\alpha\ddot{y} - \frac{1}{\gamma}\dot{y}(\alpha y + h)\right]^2 \tag{4.6}$$

where

$$\begin{aligned} f(x) &= \alpha^2(2x+3) + 6\gamma \\ g(x) &= 16\alpha\beta\gamma^2 e^{-2x} - 16\delta\gamma^3 e^{-4x} - \alpha^4(4x+3) - 12\alpha^2\gamma \\ h(x) &= 2\beta\gamma^2 e^{-2x} - \alpha^3(x+3) - 3\alpha\gamma. \end{aligned} \tag{4.7}$$

If one replaces α by $-\alpha$ and β by $-\beta$, the equation (4.6) does not change. This means $y(x;\alpha,\beta,\gamma,\delta) = y(x;-\alpha,-\beta,\gamma,\delta)$ and hence $u(z;\alpha,\beta,\gamma,\delta) = u(z;-\alpha,-\beta,\gamma,\delta)$. Let $v = v(z;\alpha,\beta,\gamma,\delta)$ and let $\bar{v} = v(z;-\alpha,-\beta,\gamma,\delta)$. Then equation (4.4) implies that

$$(v + \bar{v})[\gamma z(v - \bar{v}) + \alpha] = 0. \tag{4.8}$$

Therefore, one rederives the well known discrete Lie-point symmetry of PIII $v(z;\alpha,\beta,\gamma,\delta) = -v(z;-\alpha,-\beta,\gamma,\delta)$ [8].

Case 2. $A_2^2 + 4B_4 = 0$. In this case, the quadratic equation and (1.13) become

$$6\nu\mu z v^2 + 2(\nu z^2 u + 2\mu + \nu)v + z^2 u' + 2zu - 2\beta\nu z = 0 \tag{4.9}$$

and

$$(v')^2 = \frac{2}{z}v(\nu z v + 1)v' - \gamma v^4 + \frac{2(\mu - \nu)}{z}v^3 + uv^2 - \frac{2\beta}{z}v - \delta \tag{4.10}$$

respectively, where $\nu = \sqrt{\gamma}$ and $\mu = \alpha - 2\nu$. Suppose that $\nu \neq 0$ and let $z = r(x)$, $u = \frac{3\mu}{r}(py + q) - \frac{2\mu+\nu}{\nu r^2}$, where $r(x) = -\frac{2}{\nu}\int\frac{dx}{p(x)}$, $\dot{p} + 2q - \frac{2}{\nu r} = 0$ and $p\dot{q} + q^2 - \frac{2}{\nu r}q + \frac{4\beta}{3\mu} = 0$. Then one obtains the following second-order fourth-degree Painlevé-type equation for $y(x)$:

$$\begin{aligned} &\left\{(\ddot{y} + 2y\dot{y})^2 - 8(y+e)(\dot{y} + y^2)(\ddot{y} + 2y\dot{y}) + 16\left(y^2 + 2ey + e^2 + \frac{2\mu}{\nu^3 r^2 p^2}\right)(\dot{y} + y^2)^2\right. \\ &\qquad \left. + 16(y^4 + g_3 y^3 + g_2 y^2 + g_1 y + g_0)(\dot{y} + y^2)\right\}^2 \\ &\quad = 16(\dot{y} + y^2)\left\{(2y^2 + 2ey + f)(\ddot{y} + 2y\dot{y})\right. \\ &\qquad \left. - 4\left(\frac{\mu}{\nu^2 r p}\dot{y} + 2y^3 + h_2 y^2 + h_1 y + h_0\right)(\dot{y} + y^2)\right\}^2 \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} & e = \frac{2}{p}\left(q - \frac{1}{\nu r}\right) \qquad f = \frac{1}{2}e^2 + \frac{2}{p^2}\left(\frac{2\beta}{3\mu} - \frac{1}{\nu^2 r^2}\right) \qquad g_3 = 2\left(e - \frac{\mu}{\nu^2 r p}\right) \\ & g_2 = e^2 + f - \frac{\mu}{\nu^2 r p^2}\left(3pe + \frac{4}{\nu r}\right) \qquad g_1 = ef - \frac{\mu q}{\nu^2 r p^3}\left(3pe + \frac{2}{\nu r} + \frac{4\beta}{\mu q}\right) \\ & g_0 = \frac{1}{4}f^2 - \frac{1}{\nu^2 r p^4}(\mu p e q^2 + 4\beta q - \delta r) \qquad h_2 = 2\left(2e - \frac{\mu}{\nu^2 r p}\right) \\ & h_1 = 2e^2 + f - \frac{\mu}{\nu^2 r p^2}\left(3pe + \frac{2}{\nu r}\right) \qquad h_0 = ef - \frac{\mu q}{2\nu^2 r p^3}\left(3pe - \frac{2}{\nu r} + \frac{8\beta}{\mu q}\right). \end{aligned} \tag{4.12}$$

The one-to-one correspondence between $v(z)$ and $y(x)$ is given by (4.9) and (4.10).

Now we consider the case $\gamma = \nu = 0$. In such a case, equations (4.4) and (4.9) give the following equation for v:

$$4\alpha v + z^2 u + 2zu = 0. \tag{4.13}$$

The equations (4.5) and (4.10) give the equation

$$(zv' - v)^2 = 2\alpha z v^3 + (z^2 u + 1)v^2 - 2\beta z v - \delta z^2. \tag{4.14}$$

Suppose that $\alpha \neq 0$ and let $z^2 u = 16y - 1$, $z^2 = x$. Then solving (4.13) for v and substituting in (4.14), one obtains the following second-order second-degree equation for $y(x)$:

$$x^2 \ddot{y}^2 = -4\dot{y}^2(x\dot{y} - y) + \frac{\alpha\beta}{16}\dot{y} - \frac{\alpha^2\delta}{256}. \tag{4.15}$$

Equation (4.15) is a special case ($\lambda_1 = \lambda_2 = 0$) of the equation

$$x^2 \ddot{y}^2 = -4\dot{y}^2(x\dot{y} - y) + \lambda_1(x\dot{y} - y)^2 + \lambda_2(x\dot{y} - y) + \lambda_3\dot{y} + \lambda_4. \tag{4.16}$$

The equation (4.16) was first obtained in [3] and labelled as SDI.b and was rederived in [7].

When $\gamma = \alpha = 0$, equations (4.4), (4.9) and (4.5), (4.10) give

$$z^2 u' + 2zu = 0 \tag{4.17}$$

and

$$(zv' - v)^2 = (z^2 u + 1)v^2 - 2\beta z v - \delta z^2 \tag{4.18}$$

respectively. Equation (4.17) implies $z^2 u = k$ where k is the integration constant. Hence, (4.18) reduces to the following first-order second-degree equation:

$$(zv' - v)^2 = (k + 1)v^2 - 2\beta z v - \delta z^2. \tag{4.19}$$

Therefore, if $\alpha = \gamma = 0$, PIII admits the first integral (4.19). The transformations

$$v = -\frac{\lambda w - \lambda - \beta}{w'} \qquad w = \frac{z}{v}(v' + \lambda) \tag{4.20}$$

where $\lambda^2 = -\delta$, give one-to-one correspondence between solutions $v(z)$ of PIII and solutions $w(z)$ of the following Riccati equation:

$$2zw' + w^2 - 2w - k = 0. \tag{4.21}$$

The relation between PIII with $\alpha = \gamma = 0$ and equation (4.21) was first given in [8].

5. Painlevé IV

Let $v(z)$ be a solution of the PIV equation

$$v'' = \frac{1}{2v}(v')^2 + \frac{3}{2}v^3 + 4zv^2 + 2(z^2 - \alpha)v + \frac{\beta}{v}. \tag{5.1}$$

For PIV, Φ and Ψ have the following forms:

$$\begin{aligned} \Phi &= -\frac{1}{v}(A_2 v - A_0) \\ \Psi &= \frac{1}{v}\Big[(3 - A_2^2 - 3B_4)v^4 + \Big(8z - 2B_3 - \frac{3}{2}A_1 A_2 - A_2'\Big)v^3 \\ &\quad + (4z^2 - 4\alpha - A_1' - B_2 - \tfrac{1}{2}A_1^2 - A_0 A_2)v^2 \\ &\quad - (\tfrac{1}{2}A_0 A_1 + A_0')v + B_0 + 2\beta\Big]. \end{aligned} \tag{5.2}$$

Setting $\Phi = \Psi = 0$, one obtains $A_0 = A_2 = 0$, $A_1 = -a_1(z)$, $B_4 = 1$, $B_3 = 4z$, $B_2 = 4(z^2 - \alpha) - A_1' - \frac{1}{2}A_1^2$ and $B_0 = -2\beta$. Without loss of generality one may set $B_1 = u$. In this case, the equation $G(v) = 0$ gives

$$8A_1v^3 + 8(3zA_1 + 2)v^2 + [4B_2' + 2A_1A_1' + A_1(4B_2 + A_1^2)]v + 4u' + 2A_1u = 0. \tag{5.3}$$

Thus, one should set $A_1 = 0$ to obtain the following quadratic equation for v:

$$v^2 + 2zv + \tfrac{1}{4}u' = 0. \tag{5.4}$$

The equation (1.13) takes the form

$$(v')^2 = v^4 + 4zv^3 + 4(z^2 - \alpha)v^2 + uv - 2\beta. \tag{5.5}$$

Let $u = -y + \frac{4}{3}z^3$. Then $y(z)$ satisfies the following second-order fourth-degree equation of Painlevé type:

$$\{(y'')^2 - y'[(y')^2 - 8(z^2 + 2\alpha)y' + 16zy - f]\}^2 = 64y'\{y'' - y'(y - \tfrac{4}{3}z^3 - 8\alpha z)\}^2 \tag{5.6}$$

where $f(z) = 16(\frac{1}{3}z^4 + 4\alpha z^2 + 2\beta + 1)$.

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